

Integration of the elementary Functions (Continued)

Some useful trigonometric formulas:

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Useful integrals:

$$\int \cos^2(x) dx = \frac{x}{2} + \frac{1}{4} \sin(2x) + c$$

$$\int \sin^2(x) dx = \frac{x}{2} - \frac{1}{4} \sin(2x) + c$$

EXAMPLES

1. Substitution Technique

Using of the substitution technique for indefinite integral evaluation:

$$\begin{aligned} \int \cos(3x) dx &= \left| \begin{array}{l} z = 3x \rightarrow z' = 3 \rightarrow \frac{dz}{dx} = 3 \\ dz = 3dx \rightarrow dx = \frac{dz}{3} \end{array} \right| = \int \cos(3x) \frac{dz}{3} = \\ &= \frac{1}{3} \int \cos(z) dz = \frac{1}{3} \sin(z) + c = \frac{1}{3} \sin(3x) + c \end{aligned}$$

In general for any function $f(x)$ and $a = \text{const}$:

$$\begin{aligned} \int f(ax) dx &= \left| \begin{array}{l} z = ax \rightarrow z' = a \rightarrow \frac{dz}{dx} = a \\ dz = a dx \rightarrow dx = \frac{dz}{a} \end{array} \right| = \int f(ax) \frac{dz}{a} = \\ &= \frac{1}{a} \int f(z) dz = \frac{1}{a} \int f(ax) d(ax) \end{aligned}$$

Using of the substitution technique for definite integral evaluation:

$$\int_0^{\frac{\pi}{2}} \cos(3x) dx = \left| \begin{array}{l} z = 3x \rightarrow z' = 3 \rightarrow \frac{dz}{dx} = 3 \\ dz = 3dx \rightarrow dx = \frac{dz}{3} \\ z(0) = 3 \times 0 = 0 \\ z\left(\frac{\pi}{2}\right) = 3 \times \frac{\pi}{2} = \frac{3\pi}{2} \end{array} \right| = \int_0^{\frac{3\pi}{2}} \cos(z) \frac{dz}{3} =$$

$$\frac{1}{3} \int_0^{\frac{3\pi}{2}} \cos(z) dz = \frac{1}{3} \left(\sin(z) \Big|_0^{\frac{3\pi}{2}} \right) = \frac{1}{3} \left(\sin\left(\frac{3\pi}{2}\right) - \sin(0) \right) = \frac{1}{3} \sin\left(\frac{3\pi}{2}\right) = -\frac{1}{3}$$

2. Integration by Parts

In general formula of integration by parts can be written as:

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx$$

Using integration by parts:

$$\int_0^{\pi} x \cos^2(x) dx = \left| \begin{array}{l} f = x \rightarrow f' = 1 \\ g' = \cos^2(x) \rightarrow g = \int \cos^2(x) dx = \frac{x}{2} + \frac{1}{4} \sin(2x) \end{array} \right| =$$

$$= x \left(\frac{x}{2} + \frac{1}{4} \sin(2x) \right) \Big|_0^{\pi} - \int_0^{\pi} \left(\frac{x}{2} + \frac{1}{4} \sin(2x) \right) 1 dx =$$

$$= \pi \left(\frac{\pi}{2} + \frac{1}{4} \sin(2\pi) \right) - 0 \left(\frac{0}{2} + \frac{1}{4} \sin(0) \right) - \int_0^{\pi} \left(\frac{x}{2} + \frac{1}{4} \sin(2x) \right) dx =$$

$$= \frac{\pi^2}{2} - \int_0^{\pi} \left(\frac{x}{2} + \frac{1}{4} \sin(2x) \right) dx = \frac{\pi^2}{2} - \frac{1}{2} \int_0^{\pi} x dx - \frac{1}{4} \int_0^{\pi} \sin(2x) dx =$$

$$\frac{\pi^2}{2} - \frac{1}{2} \frac{x^2}{2} \Big|_0^{\pi} - \frac{1}{4} 0 = \frac{\pi^2}{4}$$